

# I-Love-Q

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Neutron stars are not only described by their mass and radius, but also by how fast they spin (moment of inertia) and how much they can be deformed (Love number and quadrupole moment). These depend sensitively on the star's internal structure. We find universal relations between the moment of inertia, the Love number and the quadrupole moment that are independent of the equation of state. These relations can be used to learn observationally about neutron star deformations, break degeneracies in gravitational wave observations, and test General Relativity independently of nuclear-structure.

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*Introduction.* One of largest uncertainties in nuclear physics is the relation between energy density and pressure at incredibly high densities, the so-called equation of state (EoS) of nuclear matter. The interior structure of neutron stars (NSs) depends sensitively on its EoS, and so does its exterior properties, such as its mass and radius, as well as how fast the star can spin, characterized by its moment of inertia, and how much the star can be deformed, characterized by its quadrupole moment and its tidal Love number. The astrophysical observation of NSs and their exterior properties may allow us to infer the NS EoS [1–3]. For example, the observation of X-ray bursters and low-mass X-ray binaries has allowed for the *simultaneous* determination of the mass and radius to  $\mathcal{O}(10)\%$  accuracy [2–5]. Observations of millisecond pulsars, such as the double binary J0737-3039 [6, 7], may allow for the measurement of the moment-of-inertia to the same accuracy [8, 9]. Gravitational wave (GW) observations from binary NS inspirals with second-generation ground-based detectors, such as Adv. LIGO, Adv. VIRGO and KA-GRA, may allow for the measurement of the tidal Love number [10–15].

None of these observations, however, are currently accurate enough to select between the many different EoSs that have been proposed, which then leads to degeneracies in the extraction of NS information from new observations. For example, GW observations cannot extract the individual NS spins, because these are degenerate with the NS quadrupole moment. Similarly, GWs from NS binary inspirals cannot be easily used to test General Relativity (GR), again due to degeneracies in the equation of state [16–18]. We here find a way to uniquely break these degeneracies through universal *I-Love-Q* relations, ie. analytic relations between the moment-of-inertia  $\bar{I}$ , the tidal Love number  $\bar{\lambda}^{(\text{tid})}$  and the quadrupole moment  $\bar{Q}$  that are essentially insensitive to the NS EoS.

The reason why these universal relations hold is two-fold. On the one hand, we find a strong evidence that these quantities depend most sensitively on the NS interior structure close to the crust, precisely where our EoS ignorance is minimal and realistic EoSs agree. On the other hand, the effacement principle and no hair theorem hold in GR, which state that as a body approaches

the test particle limit, the motion and the exterior field of a body does not depend on its internal structure.

These universal relations have applications in several different fields. On an observational astrophysics front, the measurement of a single member of the I-Love-Q trio would automatically provide information about the other two, even when the latter may not be accessible to observation. Such a measurement would provide all the information necessary to describe the exterior properties of tidally-deformed and slowly-rotating NSs at linear and quadratic order in spin. On a GW front, the I-Love-Q relations would break the degeneracy between the quadrupole moment and the spins in GWs emitted during binary NS inspirals. Given a GW detection from such a source with a second-generation ground based detector, such as Adv. LIGO, the GW data analysis community may then be able to measure the individual NS spins to about 0.01 in dimensionless units. On a fundamental physics front, the I-Love-Q relations will allow, for the first time, for tests of GR in the NS strong-field that are not only theory-independent, but also EoS independent. In particular, independent measurements of two members of the I-Love-Q trio would constrain violations of the strong-equivalence principle, one of the pillars of GR, orders of magnitude more effectively than Solar System constraints.

*Universal Relations.* Consider an isolated, slowly-rotating NS that is described by its mass  $M$ , the magnitude of its spin angular momentum  $J$  and angular velocity  $\Omega$ , its (spin-induced) quadrupole moment  $Q$  and its moment of inertia  $I \equiv J/\Omega$ . Let us introduce dimensionless quantities  $\bar{I} \equiv I/M^3$  and  $\bar{Q} \equiv -Q/(M^3\chi^2)$ , where  $\chi \equiv J/M^2$  is the dimensionless spin parameter. Physically,  $I$  determines how fast a body can spin, given a fixed  $J$ , while  $Q$  encodes how much a body can be quadrupolarly deformed. These quantities are determined by solving the perturbed Einstein equations in the slow-rotation limit to first and second order in spin, respectively [19, 20], given an EoS.

In the presence of a companion, a NS will also be quadrupolarly deformed. The quadrupole moment tensor  $Q_{ij}$  determines the magnitude of this deformation and it can be written as  $Q_{ij} = -\lambda^{(\text{tid})}\mathcal{E}_{ij}$ , where  $\lambda^{(\text{tid})}$  is

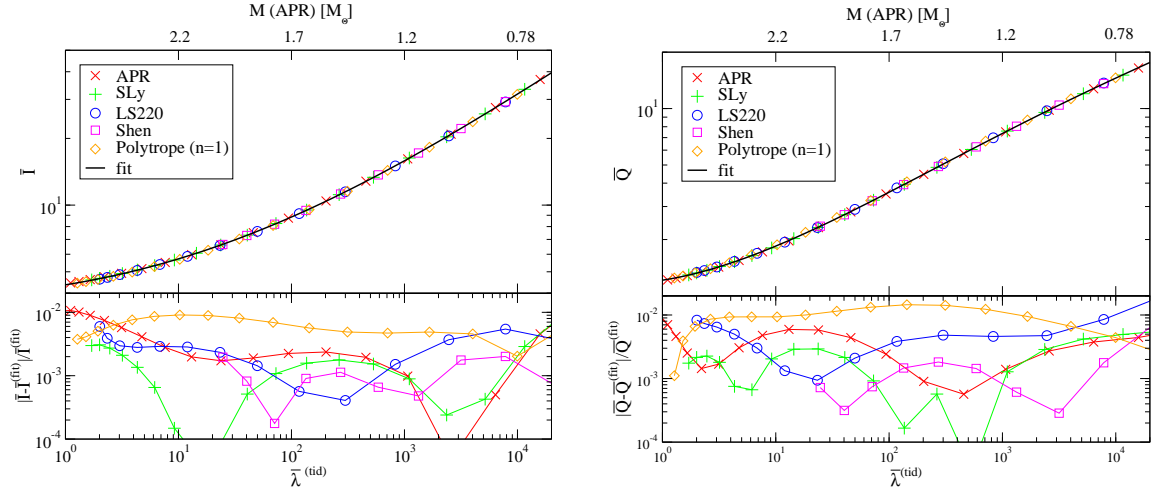


FIG. 1: (Top Left and Right) Universal I-Love and Q-Love relations for various EoSs, together with fitting curves (solid curves). On the top axis, we show the corresponding NS mass  $M$  with an APR EoS. (Bottom Left and Right) Fractional errors between the fitting curve and numerical results.

the tidal Love number and  $\mathcal{E}_{ij}$  is the quadrupole (gravitoelectric) tidal tensor that characterizes the source of the perturbation [10, 14, 20, 21]. Let us introduce the dimensionless tidal Love number  $\bar{\lambda}^{(\text{tid})} = \lambda^{(\text{tid})}/M^5$ , which physically characterizes the tidal deformability of a NS in the presence of its companion's tidal field.  $\bar{\lambda}^{(\text{tid})}$  can also be calculated by treating the tidal effect of the companion star as the perturbation to the isolated (non-rotating) NS solution [20, 21].

We here derive universal relations between  $\bar{I}$ ,  $\bar{Q}$  and  $\bar{\lambda}^{(\text{tid})}$  that are essentially insensitive to the EoS. We consider 4 different realistic EoSs: APR [22], SLy [23], Lattimer-Swesty with nuclear incompressibility of 220 MeV (LS220) [24, 25] and Shen [25–27]. For the latter two, we adopt a temperature of 0.01 MeV and an electron fraction of 30%. We also consider a simple  $n = 1$  polytropic EoS, with  $p = K\rho^{1+1/n}$ . We assume the NS is uniformly and slowly-rotating, with isotropic pressure.

The top panels of Fig. 1 present the I-Love and Q-Love relations for various EoSs. Observe that these relations hold universally, essentially independently of the EoSs. Such relations can be fitted with a polynomial on a log-log scale [20], shown here with solid black curves. The bottom panels of this figure show the fractional errors between the fitted curves and the numerical results. A similar universal relation holds between  $\bar{I}$  and  $\bar{Q}$  [20].

The I-Love-Q relations hold mainly for two reasons. One of them is that these quantities depend the most on the NS internal structure only near their crust, where our ignorance of nuclear physics is minimal and realistic EoSs agree. We verified this explicitly by computing the I-Love-Q relations for NSs with  $n = 2, 2.5$  and 3 polytropic EoSs, finding that indeed the I-Love-Q curves begin to deviate away from those shown in Fig. 1 as one increases  $n$ . The other reason is related to the effacement

principle [16, 17, 28], a consequence of the no-hair and strong equivalence principles. This principle says that the motion of a body and its gravitational field depend less and less on its internal structure as the compactness of the body, the ratio of its mass to its radius, increases. For NSs, the compactness is large enough (of order a tenth) that any possible EoS variability in the I-Love-Q relations mostly effaces away. We verified this explicitly by computing the I-Love-Q relations for NSs in a modified gravity theory where the effacement principle is violated, finding that indeed the universal relations do not hold then.

*Application to Observational Astrophysics.* Millisecond binary pulsars have the potential to measure  $I$  with 10% accuracy in the near future [8, 9]. The moment of inertia may be measurable because it induces additional precession of the periastron, as well as precession of the angular momentum vector and the NS spin vectors.

Given an observational measurement of the moment of inertia to 10%, the I-Love-Q relations automatically provide a secondary measurement of the tidal Love number and of the quadrupole moment to roughly the same accuracy. Notice, however, that the tidal Love number and the quadrupole moment would not be easily directly observable with millisecond binary pulsars. Although they do induce additional precession, their effect is suppressed relative to that of the moment of inertia, by various powers of the binary's orbital velocity to the speed of light.

*Application to GW Astrophysics.* Interferometric GW detectors are most sensitive to the GW phase of the signal. For waves emitted during NS binary inspirals, the GW phase contains a term proportional to the NS's spin-induced quadrupole moment,  $Q_1$  and  $Q_2$ , and another term proportional to its tidally-induced quadrupolar deformation  $\lambda_1^{(\text{tid})}$  and  $\lambda_2^{(\text{tid})}$ . The former enters with a factor proportional to  $v^4/c^4$  [29], while the latter en-

ters with a factor proportional to  $v^{10}/c^{10}$  [10] relative to the leading-order term. Since by Kepler's third law, the orbital velocity is related to the GW frequency via  $v \propto f^{1/3}$ , each term has a distinct frequency dependence that in principle makes them non-degenerate.

The NS quadrupole moment is degenerate with the NS's individual spins, because there is a spin-spin interaction term in the GW phase that enters at the same order in  $v/c$  as the quadrupole one [29, 30]. Such a degeneracy prevents us from independently extracting the quadrupole moment and the individual spins from a GW detection with any second- or third-generation GW detector. The Q-Love relation, however, can be used to break this degeneracy, by rewriting  $\bar{Q}$  as a function of  $\bar{\lambda}^{(\text{tid})}$ . If the Love number can be measured with a GW detection, then one can also separately measure the spins, which would be practically impossible without the Q-Love relation.

Let us then consider the accuracy to which NS parameters can be measured, given a binary NS observation. We estimate this accuracy by carrying out a Fisher analysis with two different post-Newtonian NS waveforms [20] that differ only in the choice of system parameter. In one case, we choose the parameters [15]  $\{\theta_A^i\} = (\ln \mathcal{M}, \ln \eta, \beta, D_L, t_c, \phi_c, \bar{\lambda}_s^{(\text{tid})})$ , where  $\mathcal{M} \equiv (m_1 + m_2)\eta^{3/5}$  is the chirp mass,  $\eta = m_1 m_2 / (m_1 + m_2)^2$  is the symmetric mass ratio,  $m_1$  and  $m_2$  are the component masses,  $D_L$  is the luminosity distance and  $t_c$  and  $\phi_c$  are the time and phase of coalescence. The quantity  $\beta$  is a certain combination of the individual NS spins [30], while  $\bar{\lambda}_s^{(\text{tid})}$  is the averaged, dimensionless tidal Love number. In the other case, we choose the parameters [20]  $\{\theta_B^i\} = (\ln \mathcal{M}, \delta_m, \chi_s, \chi_a, D_L, t_c, \phi_c, \bar{Q}_s(\bar{\lambda}_s^{(\text{tid})}), \bar{\lambda}_s^{(\text{tid})})$ , where  $\chi_s$  is the averaged, dimensionless NS spin,  $\chi_a$  is the difference of the dimensionless spins divided by two,  $\bar{Q}_s$  is the averaged quadrupole moment and we have used the Q-Love relation to express  $\bar{Q}_s$  as a function of  $\bar{\lambda}_s^{(\text{tid})}$ .

Figure 2 shows the measurement accuracy of different parameters, given a binary NS observation at  $D_L = 100$  Mpc with Adv. LIGO and a signal-to-noise ratio of  $\text{SNR} \approx 30$  for three different systems: (i)  $(m_1, m_2) = (1.45, 1.35)M_\odot$ ,  $\chi_1 = \chi_2$ , (ii)  $(m_1, m_2) = (1.45, 1.35)M_\odot$ ,  $\chi_1 = 2\chi_2$  and (iii)  $(m_1, m_2) = (1.4, 1.35)M_\odot$ ,  $\chi_1 = \chi_2$ . The solid red line corresponds to the measurement accuracy of  $\beta$  with waveforms parameterized by  $\theta_A$ , while the other lines correspond to the measurement accuracy of  $\chi_s$  and  $\chi_a$  with waveforms parameterized by  $\theta_B$ . Observe that  $\Delta\beta$  and  $\Delta\chi_a$  are almost identical for all systems, since they are dominated by their priors. Observe also that these errors are so large that the relative fractional errors can be larger than 100%. On the other hand, thanks to the Q-Love relation with the parameterization of  $\theta_B$ , one can determine the averaged spin  $\chi_s$  to  $\sim 0.01$ .

*Application to Fundamental Physics.* NS observations allow for tests of GR [18] when the gravitational field is much stronger than in the Solar System, where Einstein's theory has already been stringently tested [16, 17]. Un-

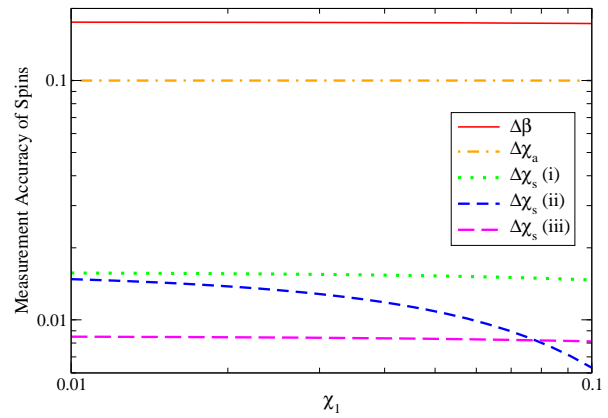


FIG. 2: Measurement accuracy of spin parameters  $\beta$ ,  $\chi_s$  and  $\chi_a$  with Adv. LIGO given a detection at a luminosity distance of 100 Mpc with  $\text{SNR} \approx 30$ . We consider three different NS binaries, labeled by (i), (ii) and (iii), as described in the text.  $\Delta\beta$  and  $\Delta\chi_a$  are almost identical among all systems, since they are dominated by their priors ( $|\beta| < 0.2$  and  $|\chi_a| < 0.1$ ).

fortunately, however, these tests are most of the time not effective because of degeneracies between modified gravity effects and the EoS. The I-Love-Q relations can be used to break this degeneracy and thus allow for model-independent and EoS-independent tests of GR with NS observations, which could be thought of as null tests of the strong-equivalence principle.

The most robust GR test would require at least two independent measurements of any two quantities in the I-Love-Q trio. Given a single measurement of any one of them, the I-Love-Q relations give us what the other two values must be in GR. The second measurement can then be used as a redundancy test. A disagreement between this measurement and the I-Love-Q predicted value would indicate a model-independent and EoS-independent GR deviation.

As an example, let us assume that a millisecond binary pulsar observation has led to a measurement of  $\bar{I}$  to 10% accuracy [8, 9], while a GW observation has led to a measurement of  $\bar{\lambda}^{(\text{tid})}$  to roughly 60% for a system with similar masses [10–15, 20]. Such a measurement would lead to an error box in the I-Love plane, centered about the measured value, as shown in the left panel of Fig. 3. Any modified gravity theory must then be such that its I-Love relation runs through this error box. This panel shows not only the I-Love relation in GR, but also this relation in a certain modified gravity theory (dynamical Chern-Simons (CS) gravity [31]) for a variety of EoSs. The requirement that the CS I-Love relation be inside of the error box automatically places a stringent constraint on this theory, which would be approximately 6 orders of magnitude stronger [20] than current Solar System constraints [32].

Even if one were not able to measure the moment of inertia in the near future, one can still test GR through the relation between the tidal Love number and the NS com-

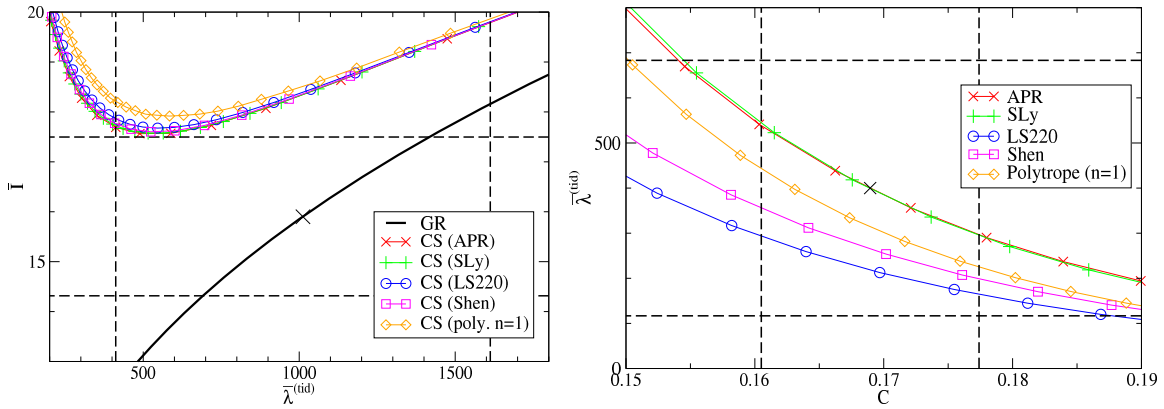


FIG. 3: (Left) Possible error box in the I-Love plane, given two independent observations of the moment of inertia and the tidal Love number, shown with a black cross. The black solid line shows the I-Love relation in GR, while all the other lines show the same relation in dynamical CS gravity. (Right) Possible error box in the Love-C plane, given two independent observations of the Love number and the NS compactness, shown with a black cross. The different curves show the Love-C relation in GR for several EoSs.

pactness  $C$ . Let us then assume that a GW observation has measured  $\bar{\lambda}^{(\text{tid})}$  to roughly 70% [20] and that future low-mass X-ray binary observations have measured  $C$  to 5% for a system with similar masses [2, 3]. Such observations would lead to an error box in the Love- $C$  plane, centered about the observed value, as shown in the right panel of Fig. 3. The Love- $C$  relation, however, is dependent on the EoS, as shown by the different EoS curves in this figure. However, the vertical distance between all curves is much smaller than the error in the measurement of the tidal Love number, making the Love- $C$  relation *effectively* EoS-independent. The requirement that any Love- $C$  relation run through this error box allows for a model-independent and an effectively EoS independent test of GR.

*Discussions.* The I-Love- $Q$  relations open the door to exciting applications in astrophysics, GW theory and fundamental physics. We have here performed a cursory study of possible applications, but these could be followed up by much more detailed analysis. For example, the measurement accuracy of GW phase parameters was here estimated via a Fisher analysis, but this could be improved through Bayesian methods [33, 34]. One could also extend these tests to systems that do not have the same masses. Millisecond binary pulsar observations may indeed not observe NS systems with the same masses

as GW observations. We have here verified, however, that all the applications discussed above are robust, even when all the NS masses differ by about 10% [20].

The analysis of the I-Love- $Q$  relations presented here opens up the road for multiple follow-up studies. For example, one could determine whether these relations hold for NSs with anisotropic pressure [35], large internal magnetic fields and rapid rotation [36–39]. One could also investigate whether there are universal relations between other quantities, such as the  $f$ - and  $w$ -modes of NS oscillations [40–42].

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